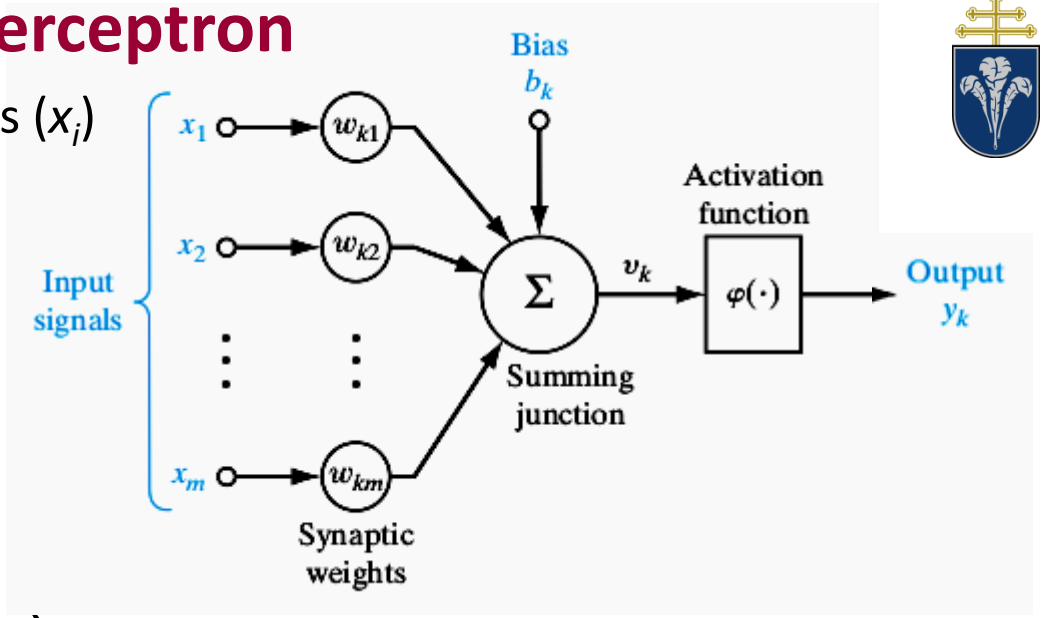
Outline

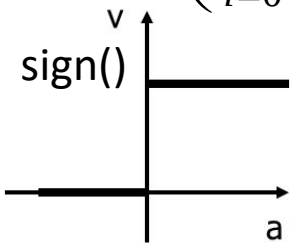
- Properties of the perceptron
- Input-output pairs
- Perceptron learning method
- Perceptron learning example
- Proof of convergence
- Good material:
  http://hagan.okstate.edu/4_Perceptron.pdf

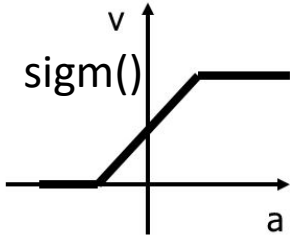
The Perceptron

- Receives input through its synapsis (x_i)
- Synapsis are weighted (w_i)
- A *b* value biases the sum to enable asymmetric behavior
- A weighted sum is calculated
- Activation function applied



$$y_k = \varphi\left(\sum_{i=0}^{m} w_{ki} x_i\right) = \varphi(\mathbf{w}^T \mathbf{x})$$

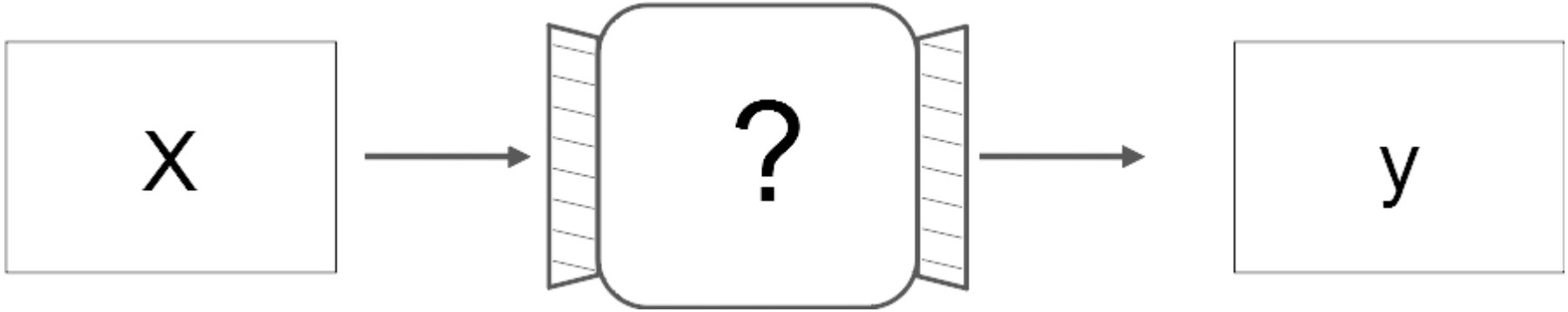




x_i : input vector
w_{ki} : weight coefficient vector of neuron k
b_k : bias value of neuron k
o_k : output value of neuron k

Neural Networks

Perceptron is an Input → Output device



As opposed to **Traditional Computers** where
- the math of the functionality is known
- the known math should be programmed

At **Neural Networks**
- the math behind the functionality is unknown
- the functionality is “illustrated” with examples

Function illustrated by examples

- Given a set of input-output pairs

  $x_j \rightarrow d_j$ (x_j: input vector; d_j: desired output)

	X		d
#	X_1	X_2	
1	0	0	0
2	0	1	0
3	1	0	0
4	1	1	1

- Number of input vectors
  - Finite/limited set (*e.g.* AND function)
    - Equivalent with a look-up-table (LUT), math known
    - Mathematically it is correct to define a function by listing all the IO pairs
    - Goal: generate a simpler than LUT decision making device through learning
  - Infinite/open set (customers of a bank asking for a loan)
    - Math behind is unknown, cannot be coded directly
    - Goal: generate the function through learning
    - It should predict well the output of a previously unknown/untested input (**GENERALIZATION**)

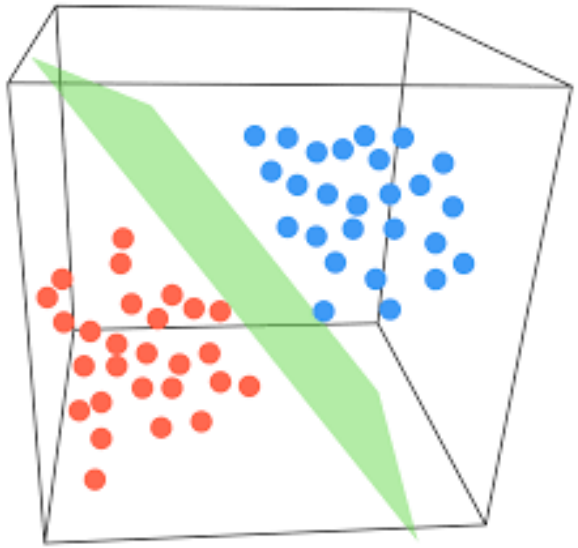
	X				d
#	age	gender	debt	salary	
1	25	M(1)	25	100	Y(1)
2	22	F(2)	18	80	Y(1)
3	65	M(1)	3000	200	N(0)
.	.	.	.	.	.
.	.	.	.	.	.

Good news: we can use the same learning/training method!!!

Linear separability

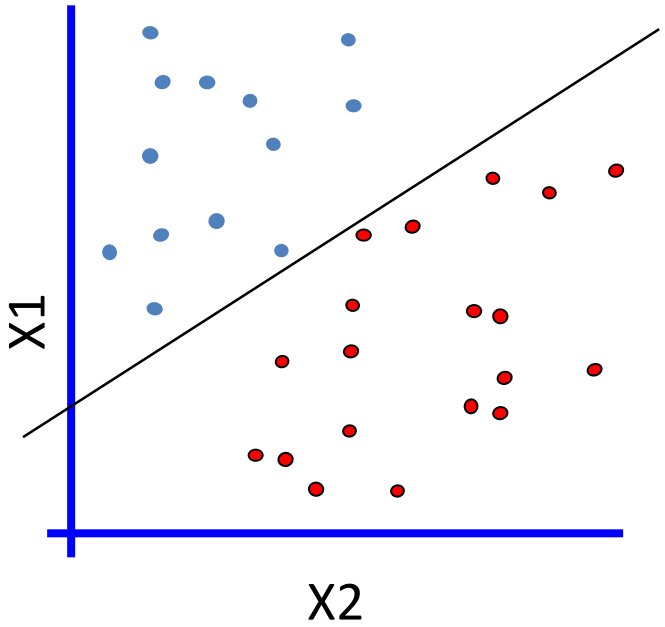
- Today, we assume that the IO sets are linearly separable
- The decision boundary is a hyperplane defined:

$$\mathbf{w}^T\mathbf{x} = 0$$



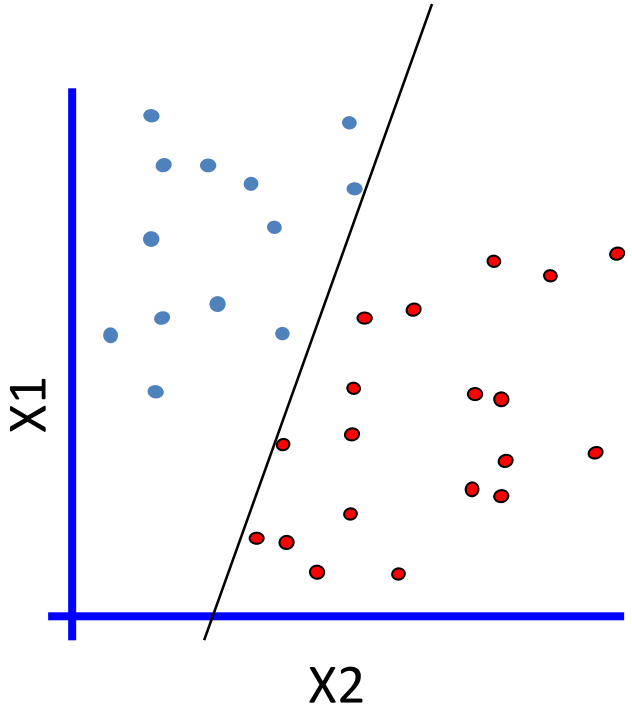
- Positive side of the hyperplane is classified: +1 (yes)
- Negative side of the hyperplane is classified : 0 (no).

Which boundary surface to use, if there are many?



How would you classify this data?

Which boundary surface to use, if there are many?



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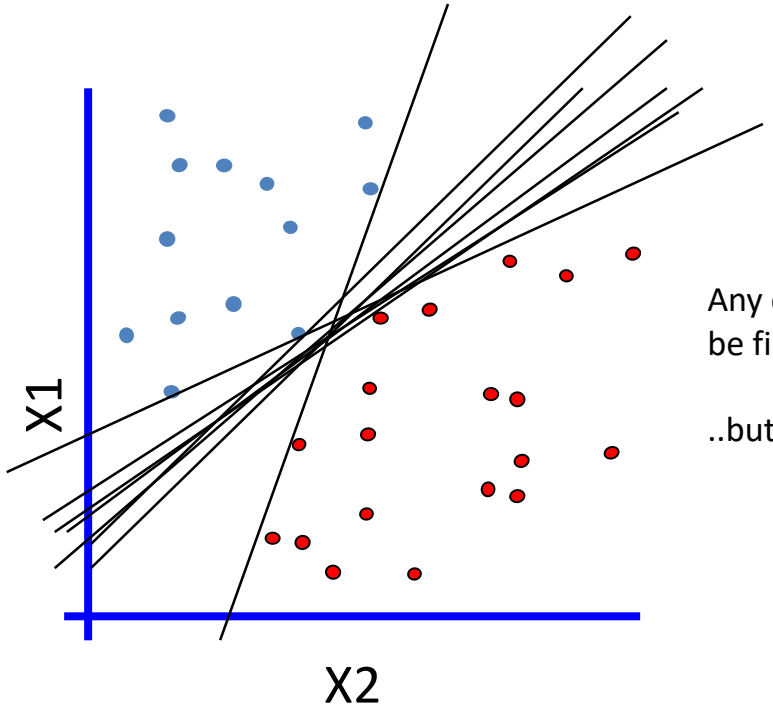
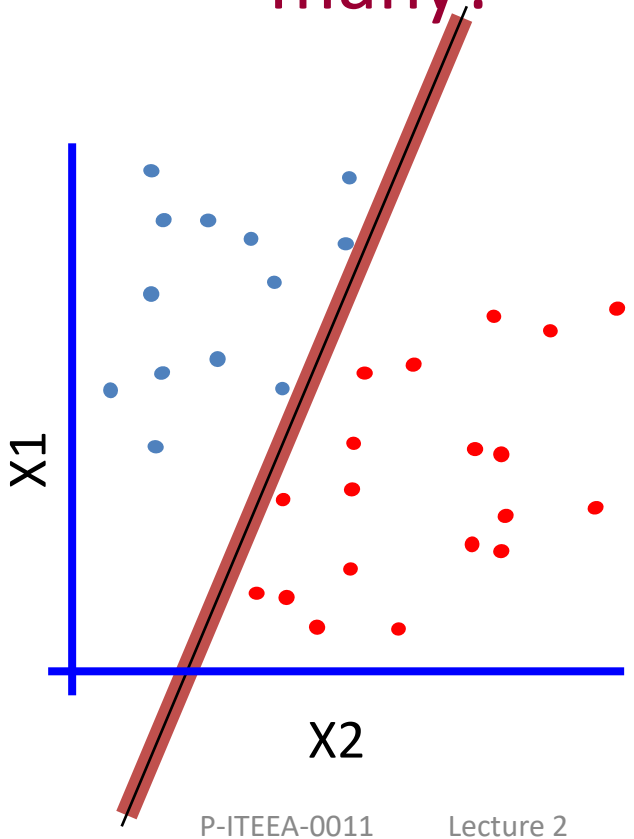


Any of these would be fine..

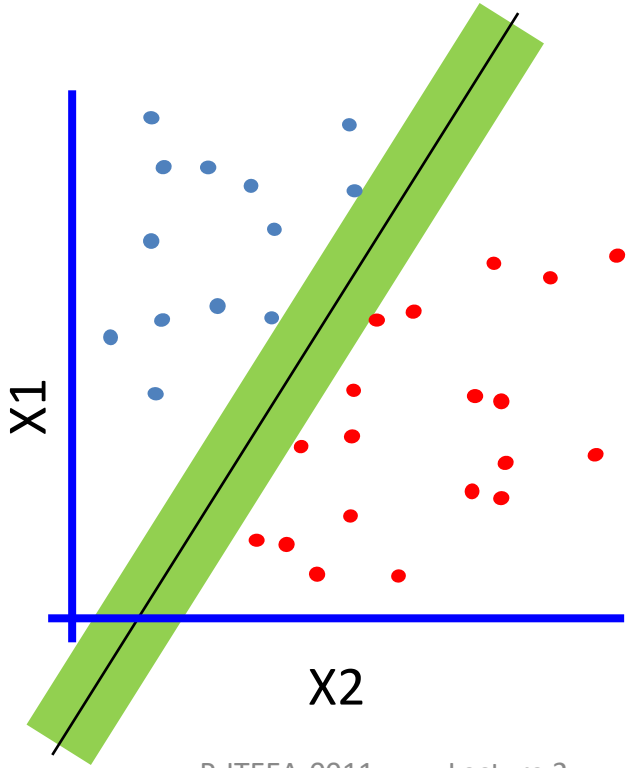
..but which is best?

Which boundary surface to use, if there are many?



Maximum Margin:
Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a data point.

Which boundary surface to use, if there are many?



Maximum Margin:
Define the margin of a linear classifier as the width that the boundary could be increased by before hitting a data point.

What does learning mean?

- Given an annotated dataset

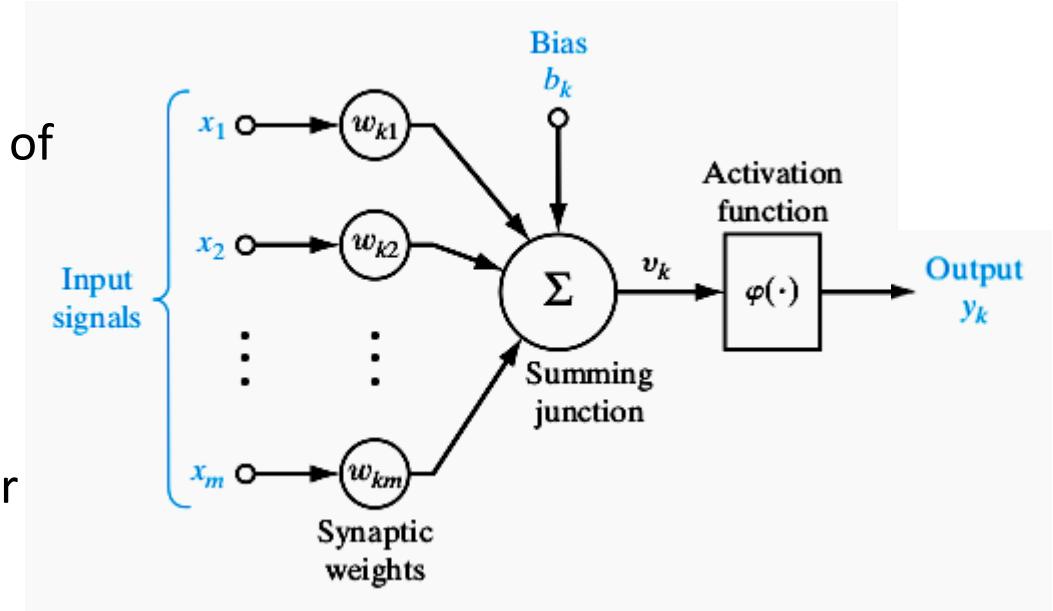
  $\mathbf{x}_j \rightarrow d_j$

- Given the parametric equation of the perceptron

  $y = sign(\mathbf{w}^T\mathbf{x})$

- Goal: find the optimal $\mathbf{w}_{opt}$ weights (parameters), where for each j

  $d_j = sign(\mathbf{w}_{opt}^T\mathbf{x}_j)$



The learning algorithm: Datasets

- Training set
  - *Set of input – desired output pairs*
  - *Will be used for training*

$$X^+ = \{\mathbf{x} : \ d = +1\}$$
$$X^- = \{\mathbf{x} : \ d = 0\}$$

- Test set
  - Used, when we have large set of input vectors (not used today)
  - *Set of input – desired output pairs*
  - *Will be used for testing and scoring the result*

- We assumed that X^+ and X^- must be linearly separable
- We are looking for an optimal parameter set:

$$X^+ = \{\mathbf{x} : \ \mathbf{w}_{\text{opt}}^{\text{T}}\mathbf{x} > 0\},$$
$$X^- = \{\mathbf{x} : \ \mathbf{w}_{\text{opt}}^{\text{T}}\mathbf{x} < 0\}.$$

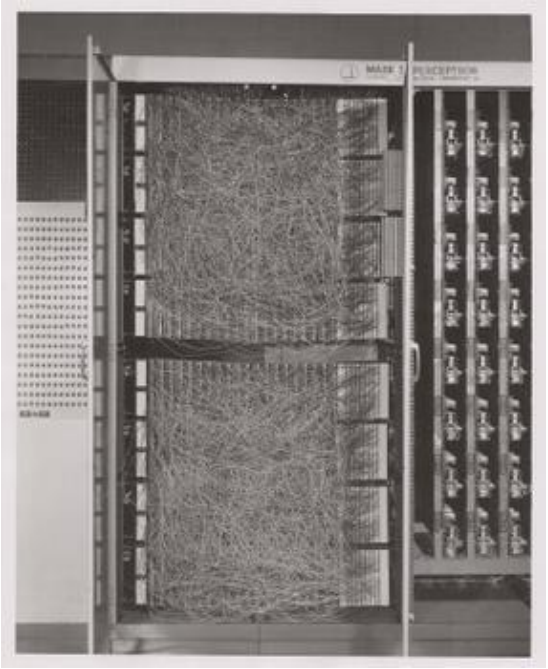
The learning algorithm: Recursive algorithm

- We have to develop a recursive algorithm called learning, which can learn the weight step by step, based on observing
  - the **(i) input,**
  - the **(ii) weight vector**,
  - the **(iii) desired output,** and
  - the **(iv) actual output** of the system.
- This can be described formally as follows:

$$\mathbf{w}(k+1) = \Psi\big(\mathbf{x}(k), \mathbf{w}(k), d(k), y(k)\big) \quad \rightarrow \quad \mathbf{w}_{opt}$$

The learning algorithm: Perceptron Learning Algorithm

- In a more ambitious way it can be called intelligent, because
  - perceptron can learn through examples (adapt),
  - even the function parameters are fully hidden.
- Perceptron learning was introduced by Frank Rosenblatt 1958
  - Built a 20x20 image sensor
  - With analog perceptron
  - 400 weights controlled by electromotors



The learning algorithm: Recursive steps

1. *Initialization.*

   Set **w**(0)=**0** or **w**(0)=**rand**

2. *Activation.*

   Select a $\mathbf{x}_k$ → d_k pair

3. *Computation of actual response*

   $$y(k) = sign\left(w^T(k)x(k)\right)$$

4. *Adaptation of the weight vector*

   $$\mathbf{w}(k+1) = \Psi\left(\mathbf{x}(k), \mathbf{w}(k), d(k), y(k)\right)$$

5. *Continuation*

   Until all responses of the perceptron are OK

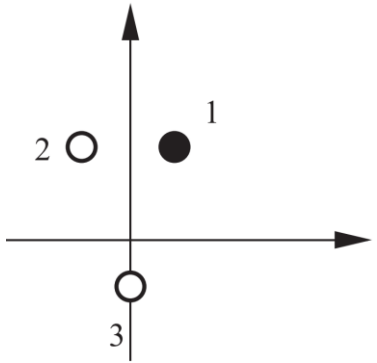
Weight update: very simple example

- Given a 3 input vector example
- Assume that bias is zero
  (decision boundary will cross the origo)
- Random initialization

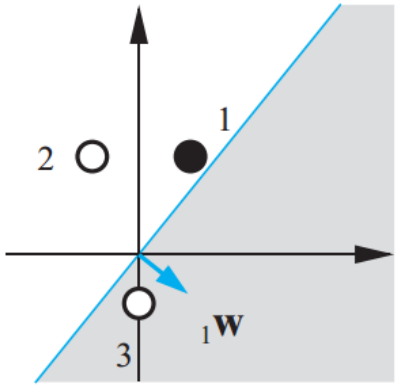
$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, d_1 = 1;$$

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, d_2 = 0;$$

$$\mathbf{x}_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, d_3 = 0;$$



$$\mathbf{w}^T(1) = [1 \quad -0.8];$$



Remember: the weight vector is orthogonal to the decision boundary!!!
Decision boundary: $x_1 - 0.8x_2 = 0$
Its orthogonal vector is: (1, -0.8)

Weight update: very simple example

- Test with the first input vector $\mathbf{x}_1 = \begin{bmatrix}1\\2\end{bmatrix}, d_1 = 1;$

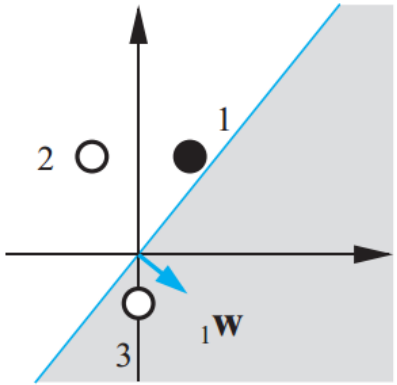
$\mathbf{w}^T(1) = [1 \quad -0.8];$

d_j-y_j > 0

$$y_1(1) = sign(w^T(1)x_1) = sign\left([1 \quad -0.8] \quad \begin{bmatrix}1\\2\end{bmatrix}\right) = sign(1 - 1.6) = 0$$

The result is not OK! Positive misclassification: Instead of 1, the result is 0!!

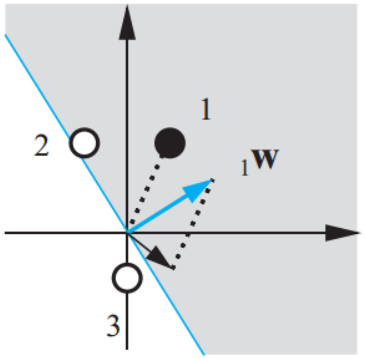
(The normal vector points to the positive side of the decision boundary.)



Idea: **add** the vector pointing to the positively misclassified point to the orthogonal vector of the decision boundary, to rotate it **towards** the point!

$w(k+1)=w(k)+x_1$

$$\mathbf{w}^T(2) = [1+1 \quad -0.8+2] = [2 \quad 1.2];$$



Weight update: very simple example

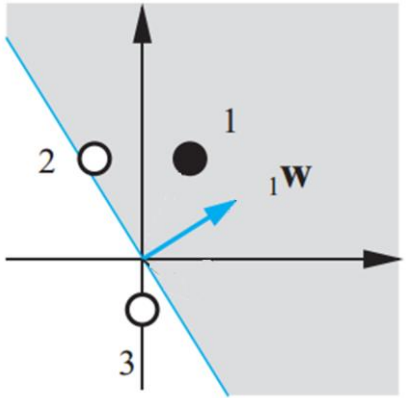
- Test with the second input vector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, d_1 = 0;$

$\mathbf{w}^T(2) = [2 \quad 1.2];$

d_j-y_j < 0

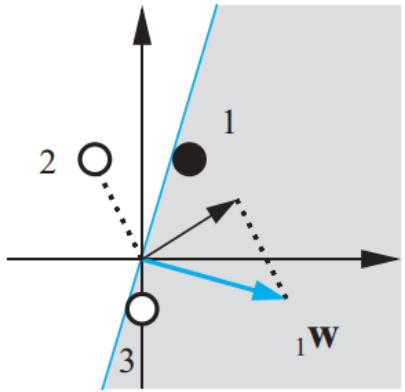
$$y_2(2) = sign(w^T(2)x_2) = sign\left([2 \quad 1.2] \quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = sign(-2 + 2.4) = 1$$

The result is not OK! Negative misclassification: Instead of 0, the result is 1!!



Idea: **subtract** the vector pointing to the negatively misclassified point to the orthogonal vector of the decision boundary, to rotate it **away** the point!
$w(k+2)=w(k+1)-x_2$

$$\mathbf{w}^T(3) = [2 - (-1) \quad 1.2 - 2] = [3 \quad -0.8]$$



Weight update: very simple example

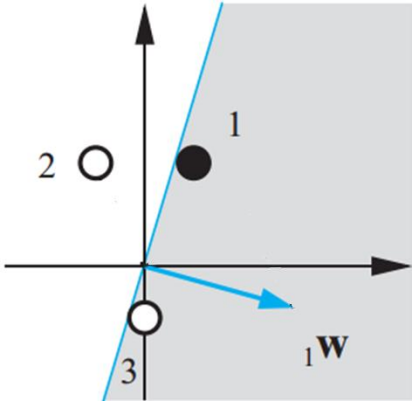
- Test with the third input vector $\mathbf{x}_3 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, d_3 = 0;$

d_j-y_j < 0

$\mathbf{w}^T(3) = [3 \quad -0.8];$

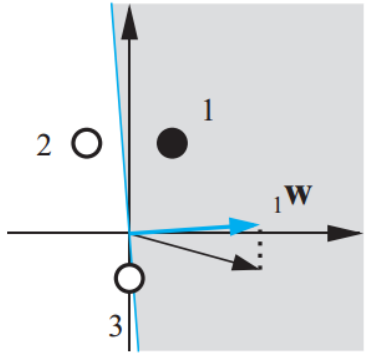
$$y_3(3) = sign(w^T(3)x_3) = sign\left([3 \quad -0.8] \quad \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right) = sign(0 + 0.8) = 1$$

The result is not OK! Negative misclassification: Instead of 0, the result is 1!!



Again: **subtract** the vector pointing to the negatively misclassified point to the orthogonal vector of the decision boundary, to rotate it **away** the point!
$w(k+3)=w(k+2)-x_3$

$$\mathbf{w}^T(4) = [3 - 0 \quad -0.8 - (-1)] = [3 \quad 0.2];$$



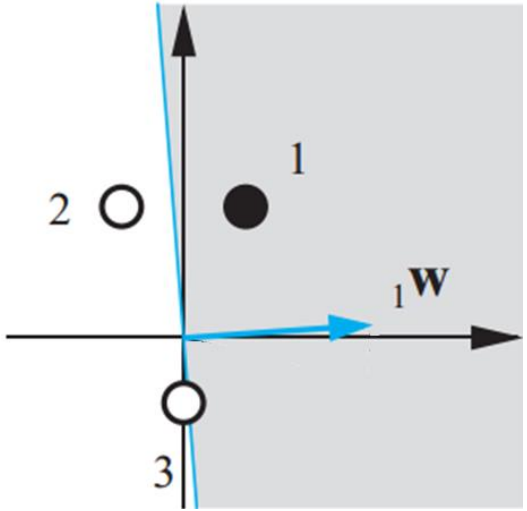
Weight update: very simple example

- Start again:
  - Test with the again with the first vector

    The result is OK!
    - Do not modify!!!
  - Test with the again with the second vector

    The result is OK!
    - Do not modify!!!
  - Test with the again with the third vector

    The result is OK!
    - Do not modify!!!
- Since all input vectors are correctly classified: we are ready



Formalization of the update rules

- Positive misclassification : ADD

  $\varepsilon = d_j - y_j = 1$ $\quad w(k+1)=w(k)+x_j$

- Negative misclassification : SUBTRACT

  $\varepsilon = d_j - y_j = -1$ $\quad w(k+1)=w(k)-x_j$

- Correct classification : DO NOTHING

  $\varepsilon = d_j - y_j = 0$ $\quad w(k+1)=w(k)$

- In general:

$$w(k+1)=w(k)+\varepsilon\, x_j$$

The learning algorithm: Adaptation

We were looking for a recursive function:

$$\mathbf{w}(k+1) = \Psi\big(\mathbf{x}(k), \mathbf{w}(k), d(k), y(k)\big)$$

In general:

$$\mathbf{w}(k+1) = \mathbf{w}(k) + \varepsilon\eta\boldsymbol{x}_j$$

where is the error function

$$\varepsilon(k) = d(k) - y(k)$$

and

$$d(k) = \begin{cases} 0 & \text{if } \mathbf{x}(k) \text{ belongs to class } \mathrm{X}^{+} \\ -1 & \text{if } \mathbf{x}(k) \text{ belongs to class } \mathrm{X}^{-} \end{cases},$$

η is the learning rate
(η controls the learning speed and should be positive)

AND

Weight update strategy

- Apply all the input vectors in one after the others, selecting them randomly
- Instance update
  - Update the weights after each input
- Batch update
  - Add up the modifications
  - Update the weights with the sum of the modifications, after all the inputs were applied
- Mini batch
  - Select a smaller batch of input vectors, and do with that as in the batch mode

Perceptron Convergence theorem (1)

Assumptions:

- ***w**(0)=0*
- *the input space is linearly separable, therefore* $\mathbf{w}_o$ *(stands for* $\mathbf{w}_{optimal}$*) exists:*

$$x \in X^+: \quad w_o^T x > 0: \; d = 1$$
$$x \in X^-: \quad w_o^T x < 0: \; d = 0$$

- *Let us denote* $\tilde{x} = -x$

$$\tilde{x} \in \tilde{X}^-: \quad w_o^T \tilde{x} > 0: \; d = 1$$

For the proof, see also: Simon Haykins: Neural Networks and Learning Machines, *Section 1.3:* http://dai.fmph.uniba.sk/courses/NN/haykin.neural-networks.3ed.2009.pdf

Perceptron Convergence theorem (2)

- Idea:
  - During the training, the network will be activated with those input vectors (one after the other), where the decision is wrong, hence non zero adaptation is needed:

$$x(j) \in X^{+}: \quad w^{T}(j)x(j) < 0, \quad y = 0, \quad d = 1$$

$$x(j) \in \tilde{X}^{-}: \quad w^{T}(j)x(j) < 0, \quad y = 0, \quad d = 1$$

  - Note: The error function is always positive ($\varepsilon = 1$)

Perceptron Convergence theorem (3)

- According to the learning method:
- $w(n+1)=w(0)+\eta x(0)+\eta x(1)+\eta x(2)+\eta x(3)+\ldots+\eta x(n)$
  - where

$$x(j)\in X^{+}:\quad w^{T}(j)x(j)<0,\quad y=-1,\ d=1$$

or

$$x(j)\in \tilde{X}^{-}:\quad w^{T}(j)x(j)<0,\quad y=-1,\ d=1$$

  - The decision boundary will be:

$$\eta w^{T}x=0$$

which means that η is a scaling factor, therefore it can be choosen for any positive number.

Let us use $\eta=1$, *therefore* $\eta\varepsilon=1$

Perceptron Convergence theorem (4)

- We will calculate $\|w(n+1)\|^2$ in two ways, and give an upper and a lower boundary, and it will turn out that an n_{max} exists, and beyond that the lower boundary is higher than the upper boundary (squeeze theorem, sandwitch lemma (*közrefogási elv, rendőr elv*))

Perceptron Convergence theorem (5)
lower limit (1)

According to the learning method, the presented input vectors are added up:

$$w(n+1) = w(0) + x(0) + x(1) + \ldots + x(n)$$

w(0)=0

Multiply it with w_o^T from the left:

$$w_o^T w(n+1) = w_o^T x(0) + w_o^T x(1) + \ldots + w_o^T x(n)$$

$$0 < \alpha \le w_o^T x(j)$$

Because each input vector (or its opposite) were selected that way.

$$0 < \alpha = \min_{x(n) \in \{X^+, \tilde{X}^-\}} w_o^T x(n)$$

$$w_o^T w(n+1) \ge n\alpha$$

Perceptron Convergence theorem (6)
lower limit (2)

$$w_o^T w(n+1) \geq n\alpha$$

We apply Cauchy Schwarty inequality $\|a\|^2 \|b\|^2 \geq \|a^T b\|^2$

$$\|w_0^T\|^2 \|w(n+1)\|^2 \geq \|w_o^T w(n+1)\|^2 \geq n^2\alpha^2$$

Lower limit:

$$\|w(n+1)\|^2 \geq \frac{n^2\alpha^2}{\|w_0^T\|^2}$$

Lower limit proportional with n^2

Perceptron Convergence theorem (7) upper limit (1)

Let us have a different synthetization approach of *w(n+1)*:

$$w(k+1) = w(k) + x(k)$$

for k= 0 … n

Squared Euclidian norm:

$$\|w(k+1)\|^2 = \|w(k)\|^2 + \|x(k)\|^2 + 2w(k)^T x(k)$$

$$w(k)^T x(k) < 0$$

Because each input vector (or its opposite) were selected that way.

$$\|w(k+1)\|^2 \leq \|w(k)\|^2 + \|x(k)\|^2$$

$$\|w(k+1)\|^2 - \|w(k)\|^2 \leq \|x(k)\|^2$$

for k= 0 … n

Perceptron Convergence theorem (8)
upper limit (2)

$$\|w(k+1)\|^2 - \|w(k)\|^2 \le \|x(k)\|^2$$

Summing up the upper term:

$$\sum_{k=0}^{n}\left(\|w(k+1)\|^2 - \|w(k)\|^2\right) \le \sum_{k=0}^{n}\|x(k)\|^2$$

Telescoping sum: $\sum_{i=1}^{n}(a_{i+1}-a_i) = a_{n+1} - a_1$

Example: $\sum_{i=1}^{4}(a_{i+1}-a_i) = \cancel{a_2} - a_1 +$
$+\cancel{a_3} - \cancel{a_2} +$
$+\cancel{a_4} - \cancel{a_3} +$
$+a_5 - \cancel{a_4} =$
$= \boldsymbol{a_5 - a_1}$

Note that there is a telescoping sum in the left hand side.

$\|w(n+1)\|^2 - \|w(0)\|^2$:

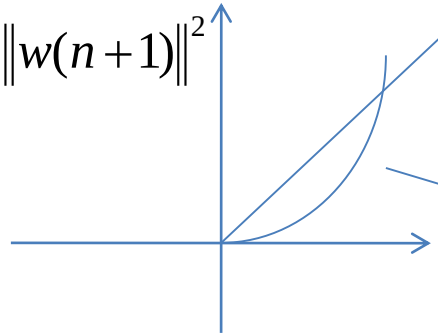
$$0 < \beta = \max_{x(k)\in\{X^+,\tilde{X}^-\}}\|x(k)\|^2$$

$$\|w(n+1)\|^2 \le (n+1)\beta$$

$\|w(0)\|^2 = 0$

Upper limit linearly proportional with n

Perceptron Convergence theorem (9) comparing upper and lower limits



$$\|w(n+1)\|^2 \leq (n+1)\beta$$

$$\|w(n+1)\|^2 \geq \frac{n^2\alpha^2}{\|w_0^T\|^2}$$

Linear upper limit and squared lower limit cannot grow unlimitedly

n_{max} should exist

$$n_{\max} = \frac{\beta\|w_0\|^2}{\alpha^2}$$